

NONNEGATIVE MORSE POLYNOMIAL FUNCTIONS AND POLYNOMIAL OPTIMIZATION

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Dedicated to Professor Hà Huy Vui on the occasion of his 65th birthday

ABSTRACT. In this paper we study the representation of Morse polynomial functions which are nonnegative on a compact basic closed semi-algebraic set in \mathbb{R}^n , and having only finitely many zeros in this set. Following C. Bivà-Ausina (Math Z 257:745–767, 2007), we introduce two classes of non-degenerate polynomials for which the algebraic sets defined by them are compact. As a consequence, we study the representation of nonnegative Morse polynomials on these kinds of non-degenerate algebraic sets. Moreover, we apply these results to study the polynomial optimization problem for Morse polynomial functions.

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1. INTRODUCTION

Let us denote by $\mathbb{R}[X]$ the ring of real polynomials in n variables x_1, \dots, x_n , and by $\sum \mathbb{R}[X]^2$ the set of all finitely many sums of squares (SOS) of polynomials in $\mathbb{R}[X]$. Let us fix a finite subset $G = \{g_1, \dots, g_m\}$ in $\mathbb{R}[X]$. Let

$$K_G = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_m(x) \geq 0\}$$

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be the *basic closed semi-algebraic set* in \mathbb{R}^n generated by G . Let

$$M_G := \left\{ \sum_{i=1}^m s_i g_i \mid s_i \in \sum \mathbb{R}[X]^2 \right\}$$

be the *quadratic module* in $\mathbb{R}[X]$ generated by G , and let

$$T_G := \left\{ \sum_{\sigma=(\sigma_1, \dots, \sigma_m) \in \{0,1\}^m} s_\sigma g_1^{\sigma_1} \cdots g_m^{\sigma_m} \mid s_\sigma \in \sum \mathbb{R}[X]^2 \right\}$$

denote the *preordering* in $\mathbb{R}[X]$ generated by G . It is clear that $M_G \subseteq T_G$, and if a polynomial belongs to T_G (or M_G) then it is nonnegative on K_G . However the converse is not always true, that means there exists a polynomial which is nonnegative on K_G but it does not belong to T_G (resp. M_G). The well-known examples (cf. [9]) are Motzkin's polynomial, Robinson's polynomial, etc. in the case $G = \emptyset$ (then $K_G = \mathbb{R}^n$ and $T_G = M_G = \sum \mathbb{R}[X]^2$).

In 1991, Schmüdgen [16] showed, that if a polynomial is positive on a compact basic closed semi-algebraic set then it belongs to the corresponding preordering. After that, Putinar ([15], 1993) showed that if a polynomial is positive on a basic closed semi-algebraic set whose associated quadratic module is Archimedean, then it belongs to that quadratic module.

If we allow the polynomial f having zeros in K_G , then the results above of Schmüdgen and Putinar are not true. Indeed, let us consider the following counter-example, which was given by G. Stengle [19]. Consider the set $G = \{(1 - x^2)^3\}$ in the ring $\mathbb{R}[x]$ of real polynomials in one variable. For this set, K_G is the closed interval $[-1, 1] \subseteq \mathbb{R}$ which is compact. The polynomial $f = 1 - x^2 \in \mathbb{R}[x]$ is nonnegative on $[-1, 1]$, and it has two zeros in $[-1, 1]$. It is not difficult to show that

$$f \notin T_G = M_G = \{s_0 + s_1(1 - x^2)^3 \mid s_0, s_1 \in \sum \mathbb{R}[x]^2\}.$$

Therefore a natural question is that *under which conditions a polynomial which is nonnegative on a basic closed semi-algebraic set belonging to the corresponding preordering (or quadratic module)?* C. Scheiderer (2003 and 2005) has given the following *local-global principles* to answer this question.

Theorem 1.1 ([17, Corollary 3.17]). *Let G, K_G and T_G be as above, and let $f \in \mathbb{R}[X]$. Assume that the following conditions hold true:*

- (1) K_G is compact;
- (2) $f \geq 0$ on K_G , and f has only finitely many zeros p_1, \dots, p_r in K_G ;
- (3) at each p_i , $f \in \widehat{T}_{p_i}$.

Then $f \in T_G$.

Here \widehat{T}_p (resp. \widehat{M}_p) denotes the preordering (resp. quadratic module) generated by T_G (resp. M_G) in the completion $\mathbb{R}[[X - p]]$ of the polynomial ring $\mathbb{R}[X]$ at the point $p \in \mathbb{R}^n$.

Theorem 1.2 ([18, Proposition 3.4]). *Let G, K_G and M_G be as above, and let $f \in \mathbb{R}[X]$. Assume that*

- (1) M_G is Archimedean;
- (2) $f \geq 0$ on K_G , and f has only finitely many zeros p_1, \dots, p_r in K_G ;
- (3) at each p_i , $f \in \widehat{M}_{p_i}$,

and at least one of the following conditions is satisfied:

- (4) $\dim \mathcal{V}(f) \leq 1$;
- (4') for every p_i , there exists a neighborhood U of p_i in \mathbb{R}^n and an element $a \in M_G$ such that $\{a \geq 0\} \cap \mathcal{V}(f) \cap U \subseteq K_G$.

Then $f \in M_G$.

Here $\mathcal{V}(f) = \{x \in \mathbb{R}^n | f(x) = 0\}$ denotes the vanishing set of f in \mathbb{R}^n .

The assumption on the compactness of the basic closed semi-algebraic set K_G or on the Archimedean property of M_G is necessary, and it is not difficult to verify (for Archimedean property of M_G , we can use, for example, Putinar's criterion [15]). However, it is complicated and hence not convenient in practice to verify that $f \in \widehat{T}_{p_i}$ (resp. $f \in \widehat{M}_{p_i}$) at each zero p_i of f in K_G . Therefore, it is necessary to give a *generic class* of polynomials which satisfies these conditions.

A smooth function $f : M \rightarrow \mathbb{R}$ on a smooth manifold M of dimension n is called a *Morse function* if all of its critical points are non-degenerate, i.e. if $p \in M$ is a critical point of f then the Hessian matrix $D^2f(p)$ of f at p is invertible.

It is well-known from Differential Topology and Singularity theory that almost smooth functions on smooth manifolds are Morse (cf. [1]). Furthermore, in Theorem 2.2 of section 2, we show that Morse polynomial functions solve the disadvantage mentioned above.

The assumption on the compactness of the basic closed semi-algebraic K_G in the theorems of Schmüdgen, Putinar and Scheiderer cannot be removed. In section 3 of this paper, following C. Bivà-Ausina [2], we introduce two classes of non-degenerate polynomials for which the algebraic sets defined by them are compact. As a consequence, we give a representation of nonnegative Morse polynomials on the non-degenerate algebraic set K_G (see Corollary 3.5 and Corollary 3.12).

In section 4 we give some applications of the representation of Morse polynomial functions on compact basic closed semi-algebraic sets. For the global polynomial optimization problem

$$f^* = \min_{x \in \mathbb{R}^n} f(x),$$

J.-B. Lasserre [4] and some other authors have given an SOS relaxation for this problem, which can be translated into an SDP. The finite convergence of the SOS relaxation depends mainly on the SOS representation of $f - f^*$ modulo the gradient ideal $I_{grad}(f)$ of f . One of the sufficient conditions for the finite convergence of the SOS relaxation is that the gradient ideal

$I_{grad}(f)$ is radical. We show in Proposition 4.1 that for Morse polynomial functions we don't need this condition. We apply this result to show in Theorem 4.3 that for Morse polynomial functions, the above SOS relaxation has a finite convergence.

For the constrained polynomial optimization problem on the basic closed semi-algebraic set K_G

$$f^* = \min_{x \in K_G} f(x),$$

one of the sufficient conditions for the finite convergence of the SOS relaxation is that the KKT ideal associated to the KKT system is of dimension zero (i.e. the corresponding complex KKT variety has only finitely many points) and radical. Then $f - f^*$ is in the KKT quadratic module (resp. KKT preordering). For Morse polynomial functions, we show in Proposition 4.6 that if K_G is compact (resp. M_G is Archimedean), and if f has only finitely many real KKT points in the interior of K_G , then $f - f^*$ belongs to T_G (resp. M_G).

J.-B. Lasserre [4] constructed a convex LMI problem in terms of the moment matrices to give a way to compute f^* in the case where the quadratic module M_G is assumed to be Archimedean. In his method, he assumed that $f - f^* \in M_G$. Applying Proposition 4.6 we can omit this assumption (see Corollary 4.8).

Notation: Throughout this paper, we denote \mathbb{R}_+ for the set of nonnegative real numbers; \mathbb{Z}_+ the set of nonnegative integers; $\mathbb{R}[[X]]$ the ring of formal power series in n variables x_1, \dots, x_n ; $\sum \mathbb{R}[X]^2$ (resp. $\sum \mathbb{R}[[X]]^2$) the set of all sums of squares (SOS) of finitely many polynomials (resp. formal power series) in $\mathbb{R}[X]$ (resp. $\mathbb{R}[[X]]$); $\mathbb{R}[[X - p]]$ the ring of formal power series in n variables $x_1 - p_1, \dots, x_n - p_n$, where $p = (p_1, \dots, p_n) \in \mathbb{R}^n$.

2. REPRESENTATION OF NONNEGATIVE MORSE POLYNOMIAL FUNCTIONS

In [9, Theorem 1.6.4] the author showed that if a real polynomial in one variable (resp. two variables) which is nonnegative in a neighborhood of $0 \in \mathbb{R}$ (resp. $(0, 0) \in \mathbb{R}^2$), then $f \in \mathbb{R}[[x_1]]^2$ (resp. $f \in \sum \mathbb{R}[[x_1, x_2]]^2$). Moreover, for $n \geq 3$, there exists always a polynomial which is nonnegative on \mathbb{R}^n but does not belong to $\sum \mathbb{R}[[X]]^2$. However, for a Morse polynomial function, we have a nice representation.

Lemma 2.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume that $f(x) \geq 0$ for every x in a neighborhood U of $0 \in \mathbb{R}^n$, and $f^{-1}(0) = \{0\}$. Then $f \in \sum \mathbb{R}[[X]]^2$.*

Proof. It follows from the assumption that $0 \in \mathbb{R}^n$ is an isolated minimal point of f . The minimality of the local minimum 0 of f implies that the Hessian matrix $D^2f(0)$ of f at 0 is positive semidefinite, i.e. all of its eigenvalues are nonnegative. On the other hand, since $0 \in \mathbb{R}^n$ is a critical point of f who is Morse, 0 is non-degenerate, i.e. the Hessian matrix $D^2f(0)$ is invertible. Therefore the matrix $D^2f(0)$ has no zero eigenvalues, i.e. all eigenvalues of

$D^2f(0)$ are positive. Thus the Hessian matrix $D^2f(0)$ of f at 0 is positive definite.

Then by a linear change of coordinates in a neighborhood of $0 \in \mathbb{R}^n$ we may assume that in a neighborhood of $0 \in \mathbb{R}^n$ the polynomial f is expressed in the following form:

$$f = x_1^2 + \cdots + x_n^2 + g,$$

where the order of g is greater than or equal to 3. For a monomial $a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in g such that $\sum \alpha_i \geq 3$ and there exists $i \in \{1, \dots, n\}$ such that $\alpha_i \geq 2$, we have

$$\epsilon_i x_i^2 + a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_i^2(\epsilon_i + a_\alpha x_1^{\alpha_1} \cdots x_i^{\alpha_i-2} \cdots x_n^{\alpha_n}) \in \mathbb{R}[[X]]^2, \quad (2.1)$$

where $0 < \epsilon_i \ll 1$. Note that the inclusion in (2.1) follows from the fact that for $g \in \mathbb{R}[[X]]$ with $g(0) > 0$ we have $g \in \mathbb{R}[[X]]^2$ (cf. [9, Proposition 1.6.2]). Therefore, by renumbering the indices if necessary, it suffices to prove that

$$h(x_1, \dots, x_m) := x_1^2 + \cdots + x_m^2 + ax_1 \cdots x_m \in \sum \mathbb{R}[[X]]^2,$$

where $a \in \mathbb{R}$ and $m \geq 3$. In fact, for each $i = 1, \dots, m$, denote $u_i := \prod_{j \neq i} x_j$. Then

$$h = \sum_{i=1}^m \left(\frac{1}{2} x_i + \frac{a}{m} u_i \right)^2 + \frac{3}{4} \sum_{i=1}^m x_i^2 - \frac{a^2}{m^2} \sum_{i=1}^m u_i^2.$$

Note that for any $b \in \mathbb{R}$ and for any $i \neq j$, similar to the argument shown above, we have

$$x_i^2 + bu_j^2 = x_i^2 \left(1 + b \frac{u_j^2}{x_i^2} \right) \in \mathbb{R}[[X]]^2.$$

Then $h \in \sum \mathbb{R}[[X]]^2$. The proof is complete. \square

Theorem 2.2 (Scheiderer's Positivstellensatz for Morse polynomials). *Let $G = \{g_1, \dots, g_m\} \subseteq \mathbb{R}[X]$, and K_G be the basic closed semi-algebraic set generated by G . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume the following conditions hold true:*

- (1) K_G is compact (resp. M_G is Archimedean);
- (2) $f \geq 0$ on K_G , and f has only finitely many zeros p_1, \dots, p_r in K_G , each lying in the interior of K_G .

Then $f \in T_G$ (resp. $f \in M_G$).

Proof. Since each p_i is an isolated minimal point of f , it follows from Lemma 2.1 that $f \in \sum \mathbb{R}[[X - p_i]]^2$. On the other hand, for every $i = 1, \dots, r$, p_i is an interior point of K_G , hence the condition (4') in Theorem 1.2 is fulfilled and by Lemma 2.3 below, we have

$$\widehat{T}_{p_i} = \widehat{M}_{p_i} = \sum \mathbb{R}[[X - p_i]]^2.$$

The theorem now follows from Theorem 1.1 and Theorem 1.2. \square

Lemma 2.3 ([7, Lemma 2.1]). *If $p \in K_G$ is an interior point then*

$$\sum \mathbb{R}[[X - p]]^2 = \widehat{T}_p = \widehat{M}_p.$$

Proof. Since p is an interior point of K_G , we have $g_i(p) > 0$, for all $i = 1, \dots, m$. Then $g_i \in \mathbb{R}[[X - p]]^2$ for all $i = 1, \dots, m$ (cf. [9, Proposition 1.6.2]). It follows that $\widehat{M}_p \subseteq \widehat{T}_p \subseteq \sum \mathbb{R}[[X - p]]^2$. It is clear that $\sum \mathbb{R}[[X - p]]^2 \subseteq \widehat{M}_p$. Thus we have the equalities. \square

Remark 2.4. (1) The assumption that each zero of f belongs to the interior of K_G in Theorem 2.2 is necessary. Indeed, let us consider again $G = \{(1 - x^2)^3\} \subseteq \mathbb{R}[x]$ and $f = 1 - x^2 \in \mathbb{R}[x]$. We see that $K_G = [-1, 1]$ is compact, f is a Morse function (it has a critical point at $0 \in \mathbb{R}$ and the second derivative of f at 0 is equal to -2 which is non-zero), $f \geq 0$ on K_G , and f has only two zeros in K_G . However, $f \notin T_G = M_G$. In this case, note that the zeros of f belong to the boundary of K_G .

(2) The space of Morse functions on \mathbb{R}^n is a *dense* subset of the space of all smooth functions on \mathbb{R}^n in the uniform topology (cf. [1, Theorem 5.27]). Therefore Theorem 2.2 holds for a generic class of polynomial functions on \mathbb{R}^n .

(3) In [5, Theorem 2.33], J.-B. Lasserre has given a similar Positivstellensatz for the case where f is *strictly convex* and g_j is *concave* for every $j = 1, \dots, m$.

3. REPRESENTATION OF MORSE POLYNOMIAL FUNCTIONS ON NON-DEGENERATE ALGEBRAIC SETS

As we have seen in the previous section, the compactness of the semi-algebraic sets K_G generated by a finite subset G in $\mathbb{R}[X]$ is very important for the representation of a nonnegative polynomial function on K_G . In this section we give a good class of polynomials in $\mathbb{R}[X]$ for which the algebraic sets defined by them are compact. For this purpose, we introduce the *non-degeneracy conditions* which was studied by C. Bivià-Ausina [2].

Definition 3.1 ([2], [3]). A subset $\tilde{\Gamma} \subseteq \mathbb{R}_+^n$ is said to be a *Newton polyhedron at infinity*, or a *global Newton polyhedron*, if there exists some finite subset A of \mathbb{Z}_+^n such that $\tilde{\Gamma}$ is equal to the convex hull of $A \cup \{0\}$ in \mathbb{R}^n . $\tilde{\Gamma}$ is said to be *convenient*, if it intersects each coordinate axis in a point different from the origin.

For $w \in \mathbb{R}^n$, denote

$$m(w, \tilde{\Gamma}) := \max\{\langle w, \alpha \rangle \mid \alpha \in \tilde{\Gamma}\};$$

$$\Delta(w, \tilde{\Gamma}) := \{\alpha \in \tilde{\Gamma} \mid \langle w, \alpha \rangle = m(w, \tilde{\Gamma})\}.$$

A set $\Delta \subseteq \mathbb{R}^n$ is called a face of $\tilde{\Gamma}$ if there exists some $w \in \mathbb{R}^n$ such that $\Delta = \Delta(w, \tilde{\Gamma})$. In this case, the face $\Delta(w, \tilde{\Gamma})$ is said to be *supported* by w .

Let $f = \sum_{\alpha} f_{\alpha} X^{\alpha} \in \mathbb{R}[X]$ be a polynomial and $w \in \mathbb{R}^n$. The set $\text{supp}(f) := \{\alpha \in \mathbb{N}^n \mid f_{\alpha} \neq 0\}$ is called the *support* of f . Denote

$$m(w, f) := \max\{\langle w, k \rangle \mid k \in \text{supp}(f)\};$$

$$\Delta(w, f) := \{k \in \text{supp}(f) \mid \langle w, k \rangle = m(w, f)\}.$$

The convex hull in \mathbb{R}_+^n of the set $\text{supp}(f) \cup \{0\}$ is called the *Newton polyhedron at infinity* of f and denoted by $\tilde{\Gamma}(f)$. We say that f is *convenient* if $\tilde{\Gamma}(f)$ is convenient.

The polynomial $f_w := f_{\Delta(w,f)} := \sum_{\alpha \in \Delta(w,f)} f_\alpha X^\alpha$ is called the *principal part of f at infinity* with respect to w (or $\Delta(w, f)$). For a finite subset W of \mathbb{R}^n , the *principal part of f with respect to W at infinity* is defined to be the polynomial $f_W := \sum_{\alpha \in \cap_{w \in W} \Delta(w,f)} f_\alpha X^\alpha$. If $\cap_{w \in W} \Delta(w, f) = \emptyset$, we set $f_W = 0$.

Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map. Then the convex hull of $\tilde{\Gamma}(f_1) \cup \dots \cup \tilde{\Gamma}(f_m)$ is called the *Newton polyhedron at infinity of F* and denoted by $\tilde{\Gamma}(F)$. For $w \in \mathbb{R}^n$, the *principal part of F with respect to w at infinity* is defined to be the polynomial map

$$F_w := ((f_1)_w, \dots, (f_m)_w).$$

Definition 3.2 ([2]). Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map. Denote

$$\mathbb{R}_0^n := \{w = (w_1, \dots, w_n) \in \mathbb{R}^n \mid \max_{i=1, \dots, n} w_i > 0\}.$$

We say that F is *non-degenerate at infinity* if and only if for any $w \in \mathbb{R}_0^n$, the system of equations

$$(f_1)_w(x) = \dots = (f_m)_w(x) = 0$$

has no solutions in $(\mathbb{R} \setminus \{0\})^n$.

Remark 3.3 ([2]). (1) If some component of F is a monomial, then F is automatically non-degenerate at infinity.

(2) Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $\tilde{\Gamma}(f_i)$ is convenient for every $i = 1, \dots, m$. Let $\tilde{\Gamma}_\infty(f_i)$ denote the *Newton boundary at infinity* of f_i , i.e. the union of all faces of $\tilde{\Gamma}(f_i)$ which do not passing through the origin. Then F is non-degenerate at infinity if either some component f_i is a monomial or the polygons of the family $\{\tilde{\Gamma}_\infty(f_1), \dots, \tilde{\Gamma}_\infty(f_m)\}$ verify that no segment of $\tilde{\Gamma}_\infty(f_i)$ is parallel to some segment of $\tilde{\Gamma}_\infty(f_j)$ for all $i, j \in \{1, \dots, m\}$, $i \neq j$.

Theorem 3.4 ([2, Theorem 3.8]). Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map such that f_i is convenient for all $i = 1, \dots, m$. If F is non-degenerate at infinity, then $F^{-1}(0)$ is compact.

The algebraic set $K_G := \{x \in \mathbb{R}^n \mid g_1(x) = \dots = g_m(x) = 0\}$, $g_i \in \mathbb{R}[X]$ for all $i = 1, \dots, m$, is called *non-degenerate at infinity* if the polynomial map $(g_1, \dots, g_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is non-degenerate at infinity. Then we have the following special case of Theorem 2.2.

Corollary 3.5. Let $G = \{g_1, \dots, g_m\}$ be a finite subset of $\mathbb{R}[X]$ and $K_G := \{x \in \mathbb{R}^n \mid g_i(x) = 0, i = 1, \dots, m\}$ the algebraic set defined by G . Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume the following conditions hold true:

- (1) K_G is non-degenerate at infinity and each g_i is convenient;
- (2) $f \geq 0$ on K_G , and f has only finitely many zeros p_1, \dots, p_r in K_G , each lying in the interior of K_G .

Then $f \in T_G$.

Proof. The proof follows from Theorem 3.4 and Theorem 2.2. \square

Remark 3.6. In Theorem 3.4 we need the convenience of each component f_i of the polynomial map $F = (f_1, \dots, f_m)$ for $F^{-1}(0)$ to be compact. In the following we introduce another condition of non-degeneracy for which the assumption on the convenience of each f_i can be relaxed.

Definition 3.7 ([2]). Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0)$ be a real analytic function. Suppose that the Taylor expansion of f around the origin is given by the expression $f = \sum_{\alpha} f_{\alpha} X^{\alpha}$. The set $\text{supp}(f) := \{\alpha \in \mathbb{N}^n \mid f_{\alpha} \neq 0\}$ is called the *support* of f . For a vector $v \in \mathbb{R}_+^n$, denote

$$l(v, f) := \min\{\langle v, \alpha \rangle \mid \alpha \in \text{supp}(f)\}.$$

For a finite set V of \mathbb{R}_+^n , the *local principal part of f with respect to V* is defined to be the polynomial

$$f_V := \sum_{\langle \alpha, v \rangle = l(v, f), \forall v \in V} f_{\alpha} X^{\alpha}.$$

If no such terms exist we define $f_V = 0$.

The *local Newton polyhedron of f* , denoted by $\Gamma(f)$, is the convex hull of the set

$$\bigcup_{\alpha \in \text{supp}(f)} \{\alpha + \mathbb{R}_+^n\}.$$

A subset Γ of \mathbb{R}_+^n is said to be a *local Newton polyhedron* if there exists some real analytic function f such that $\Gamma = \Gamma(f)$.

Let Γ be a local Newton polyhedron in \mathbb{R}_+^n . For $v \in \mathbb{R}_+^n$, we define

$$l(v, \Gamma) := \min\{\langle v, \alpha \rangle \mid \alpha \in \Gamma\};$$

$$\Delta(v, \Gamma) := \{\alpha \in \Gamma \mid \langle v, \alpha \rangle = l(v, \Gamma)\}.$$

A set $\Delta \subseteq \mathbb{R}_+^n$ is called a *face* of Γ if there exists some $v \in \mathbb{R}_+^n$ such that $\Delta = \Delta(v, \Gamma)$. Then we say that the vector v *supports* the face Δ .

A vector $w \in \mathbb{Z}^n$ is called *primitive* if $w \neq 0$ and it has smallest length among all vectors in \mathbb{Z}^n of the form $\lambda w, \lambda > 0$. Denote by $\mathcal{F}(\Gamma)$ the family of primitive vectors supporting some face of Γ of dimension $n - 1$.

Definition 3.8 ([2]). Let Γ be a local Newton polyhedron in \mathbb{R}_+^n . Let $f = (f_1, \dots, f_m) : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^m, 0)$ be an analytic map germ. f is said to be *adapted to Γ* if for all $V \subseteq \mathcal{F}(\Gamma)$ such that $\bigcap_{v \in V} \Delta(v, \Gamma)$ is a compact face of Γ , the system of equations

$$(f_1)_V(x) = \dots = (f_m)_V(x) = 0$$

has no solutions in $(\mathbb{R} \setminus \{0\})^n$.

Definition 3.9 ([2]). For $I \subseteq \{1, \dots, n\}$, denote

$$\mathbb{R}_I^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n | x_i = 0 \text{ for all } i \in I\}.$$

If $I = \emptyset$ then it is clear that $\mathbb{R}_I^n = \mathbb{R}^n$.

For a polynomial $f = \sum_{\alpha} f_{\alpha} X^{\alpha} \in \mathbb{R}[X]$, denote

$$f_I := \sum_{\alpha \in \mathbb{R}_I^n} f_{\alpha} X^{\alpha}.$$

If $\text{supp}(f) \cap \mathbb{R}_I^n = \emptyset$, we set $f_I = 0$. We regard f_I as a polynomial in on the variables x_i such that $i \notin I$, i.e. f_I can be regarded as the function $f_I : \mathbb{R}^{n-|I|} \rightarrow \mathbb{R}$. For a polynomial map $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$, F_I denotes the map $((f_1)_I, \dots, (f_m)_I) : \mathbb{R}^{n-|I|} \rightarrow \mathbb{R}^m$.

Let $\tilde{\Gamma}$ be a *fixed convenient Newton polyhedron at infinity* in \mathbb{R}^n and $I \subseteq \{1, \dots, n\}$. Denote by $(\tilde{\Gamma})_I$ the image of the intersection $\tilde{\Gamma} \cap \mathbb{R}_I^n$ in $\mathbb{R}^{n-|I|}$. Set

$$M := \max_{\alpha = (\alpha_1, \dots, \alpha_n) \in \tilde{\Gamma}} \{|\alpha| := \alpha_1 + \dots + \alpha_n\}.$$

Let $V_{\tilde{\Gamma}}$ denote the set of all vertices of $\tilde{\Gamma}$, and $\rho := \sum_{\alpha \in V_{\tilde{\Gamma}}} X^{\alpha}$. For any polynomial $h = \sum_{\alpha} h_{\alpha} X^{\alpha} \in \mathbb{R}[X]$, denote

$$G_M(h) := \sum_{\alpha} h_{\alpha} X^{\alpha} \|x\|^{2(M-|\alpha|)}.$$

Then we define the convenient local Newton polyhedron associated to the global Newton polyhedron $\tilde{\Gamma}$:

$$\mathbf{G}(\tilde{\Gamma}) := \Gamma(G_M(\rho)).$$

Definition 3.10 ([2]). Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map. We say that F is *globally adapted* to $\tilde{\Gamma}$ (or, *g-adapted* to $\tilde{\Gamma}$) if for any $W \subseteq \{\mathbf{w}(v) | v \in \mathcal{F}(\mathbf{G}(\tilde{\Gamma}))\}$ such that $\cap_{w \in W} \Delta(w, \tilde{\Gamma})$ is a face of $\tilde{\Gamma}$ not containing the origin, the system of equations

$$(f_1)_W(x) = \dots = (f_m)_W(x) = 0$$

has no solutions in $(\mathbb{R} \setminus \{0\})^n$. Here, for a vector $v = (v_1, \dots, v_n) \in \mathbb{R}^n$, $\mathbf{w}(v) := 2\mathbf{c} \min_i v_i - v$, where $\mathbf{c} := \mathbf{e}_1 + \dots + \mathbf{e}_n = (1, \dots, 1) \in \mathbb{R}^n$.

We say that F is *strongly g-adapted* to $\tilde{\Gamma}$ if for any $I \subseteq \{1, \dots, n\}$, $|I| \neq n$, the map $F_I : \mathbb{R}^{n-|I|} \rightarrow \mathbb{R}^m$ is g-adapted to the Newton polyhedron $(\tilde{\Gamma})_I$.

It follows from the above definition that if F is strongly g-adapted to a given convenient Newton polyhedron at infinity then $\tilde{\Gamma}(F)$ is convenient.

Theorem 3.11 ([2, Theorem 5.9]). *Let $\tilde{\Gamma}$ be a convenient Newton polyhedron at infinity. Let $F = (f_1, \dots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a polynomial map with degree $d := \max\{\deg(f_1), \dots, \deg(f_m)\}$ such that $M \geq d$. If F is strongly g-adapted to $\tilde{\Gamma}$ then $F^{-1}(0)$ is compact.*

Corollary 3.12. *Let $G = \{g_1, \dots, g_m\}$ be a finite subset of $\mathbb{R}[X]$ and $K_G := \{x \in \mathbb{R}^n | g_i(x) = 0, i = 1, \dots, m\}$ the algebraic set defined by G . Let $\tilde{\Gamma}$ be a convenient Newton polyhedron at infinity. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume the following conditions hold true:*

- (1) *The polynomial map $(g_1, \dots, g_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ has degree $\leq M$ and strongly g -adapted to $\tilde{\Gamma}$;*
- (2) *$f \geq 0$ on K_G , and f has only finitely many zeros p_1, \dots, p_r in K_G , each lying in the interior of K_G .*

Then $f \in T_G$.

Proof. The proof follows from Theorem 3.11 and Theorem 2.2. \square

4. APPLICATIONS IN POLYNOMIAL OPTIMIZATION

4.1. Unconstrained polynomial Optimization. In this section we consider the global optimization problem

$$f^* = \min_{x \in \mathbb{R}^n} f(x), \quad (4.1)$$

where $f \in \mathbb{R}[X]$ be a polynomial in n variables x_1, \dots, x_n .

It is well-known (cf. [12]) that if the gradient ideal $I_{grad}(f)$ is radical and if f attains its minimum value f^* on \mathbb{R}^n , then $f - f^*$ is SOS modulo $I_{grad}(f)$. In general we have $f - f^*$ is SOS modulo the radical $\sqrt{I_{grad}(f)}$ of the gradient ideal $I_{grad}(f)$ (cf. [12]). However, for Morse polynomial functions we have a nice representation of $f - f^*$.

Proposition 4.1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume that f achieves a minimum value f^* on \mathbb{R}^n . Then*

$$f - f^* \in \sum \mathbb{R}[X]^2 + I_{grad}(f),$$

where $I_{grad}(f) = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle$ denotes the gradient ideal of f .

Proof. Let $x^* \in \mathbb{R}^n$ be a global minimizer of f on \mathbb{R}^n . Then x^* is a critical point of f , therefore the Hessian matrix $D^2 f(x^*)$ is invertible because f is Morse. Moreover, $D^2 f(x^*)$ is positive semidefinite because x^* is a global minimizer. It follows that $D^2 f(x^*)$ is positive definite. Now apply [10, Theorem 2.1], we have

$$f - f^* \in \sum \mathbb{R}[X]^2 + I_{grad}(f).$$

\square

The following result gives degree bounds to accompany Proposition 4.1.

Proposition 4.2. *Given a positive integer d . Then there exists a positive integer l such that for each Morse polynomial function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of degree $\leq d$, if f achieves a minimum value f^* on \mathbb{R}^n , then*

$$f - f^* = \sigma + \sum_{i=1}^n h_i \frac{\partial f}{\partial x_i},$$

where $\sigma \in \sum \mathbb{R}[X]^2$ and $h_1, \dots, h_n \in \mathbb{R}[X]$, have degree bounded by l .

Proof. Similar to the proof of Proposition 4.1, if f is Morse and x^* is a global minimizer of f on \mathbb{R}^n then the Hessian matrix $D^2f(x^*)$ is positive definite. Then the proposition follows from [10, Corollary 2.4]. \square

Let $\mathbb{R}[X]_m$ denote the $\binom{n+m}{m}$ -dimensional vector space of polynomials of degree at most m . Since the gradient is zero at global minimizers, we consider the SOS relaxation:

$$f_{N,grad}^* := \max \gamma \tag{4.2}$$

$$\text{subject to } f - \gamma - \sum_{i=1}^n \phi_i \frac{\partial f}{\partial x_i} \in \sum \mathbb{R}[X]^2 \text{ and } \phi_i \in \mathbb{R}[X]_{2N-d+1}.$$

Here d is the degree of the polynomial $f \in \mathbb{R}[X]$, and N is an integer to be chosen by the user.

It is well-known (cf. [4], [6], [8], [12], [14]) that the problem (4.2) can be translated into an SDP. Moreover, $f_{N,grad}^*$ is a lower bound for f^* , and the lower bound gets better as N increases:

$$\dots \leq f_{N-1,grad}^* \leq f_{N,grad}^* \leq f_{N+1,grad}^* \leq \dots \leq f^*.$$

In the following we apply Proposition 4.1 to show the finite convergence of the relaxation given above in the case where f is a Morse polynomial function.

Theorem 4.3. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume that f achieves a minimum value f^* on \mathbb{R}^n . Then there exists an integer N such that $f_{N,grad}^* = f^*$.*

Proof. It follows from Proposition 4.1 that $f - f^*$ is SOS modulo $I_{grad}(f)$. Then by Proposition 4.2, there exists some positive integer N such that $f_{N,grad}^* \geq f^*$. Moreover, we have always that $f_{N,grad}^* \leq f^*$. Hence $f_{N,grad}^* = f^*$. \square

Remark 4.4. (1) The assumption that f achieves a minimum value f^* on \mathbb{R}^n is necessary. Indeed, let us consider the polynomial $f(x) = x^3$ in one variable. It is clear that $f^* = -\infty$ on \mathbb{R} . Moreover, we have

$$f(x) = \frac{x}{3} f'(x),$$

hence f belongs to its gradient ideal $I_{grad}(f) = \langle f' \rangle$. Therefore for every $N \geq 1$ we have $f_{N,grad}^* = 0 > f^*$.

(2) There is a generic class of polynomials which achieve their minimum values on \mathbb{R}^n . For example, in [3, Theorem 1.1] the authors showed that if $f \in \mathbb{R}[X]$ is bounded from below, convenient¹ and (Khovanskii) non-degenerate

¹The polynomial function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *convenient* if its *Newton polyhedron at infinity* $\tilde{\Gamma}(f)$ intersects each coordinate axis in a point different from the origin, that is, if for any $i \in \{1, \dots, n\}$ there exists some integer $m_i > 0$ such that $m_i \mathbf{e}_i \in \tilde{\Gamma}(f)$. Here $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ denotes the canonical basis in \mathbb{R}^n .

at infinity², then f attains its minimum value f^* on \mathbb{R}^n . Therefore we have the following consequence of this fact and Theorem 4.3.

Corollary 4.5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function which is bounded from below, convenient and (Khovanskii) non-degenerate at infinity. Then f achieves its minimum value f^* on \mathbb{R}^n , moreover, there exists an integer N such that $f_{N,grad}^* = f^*$.*

4.2. Constrained polynomial optimization. Let $G = \{g_1, \dots, g_m\}$ be a finite subset of $\mathbb{R}[X]$ and $K_G = \{x \in \mathbb{R}^n | g_i(x) \geq 0, \forall i = 1, \dots, m\}$ the basic closed semi-algebraic set generated by G . In this section we consider the following optimization problem

$$f^* := \min_{x \in K_G} f(x), \quad (4.3)$$

where $f \in \mathbb{R}[X]$ be a polynomial in n variables x_1, \dots, x_n . The *KKT system* associated to this optimization problem is

$$\begin{aligned} \nabla f - \sum_{j=1}^m \lambda_j \nabla g_j &= 0 \\ g_j &\geq 0, \quad \lambda_j g_j \geq 0, \quad j = 1, \dots, m \end{aligned} \quad (4.4)$$

where the variables $\lambda := (\lambda_1, \dots, \lambda_m)$ are called *Lagrange multipliers* and ∇f denotes the vector of partial derivatives of f . A point is called a *KKT point* if the KKT system holds at this point. Under certain regularity conditions, for example if the gradients ∇g_i of the g_i 's are linearly independent (cf. [13]), each global minimizer of f on K_G is a KKT point.

For each $i = 1, \dots, n$, denote $L_i := \frac{\partial f}{\partial x_i} - \sum_{j=1}^m \lambda_j \frac{\partial g_j}{\partial x_i}$, $i = 1, \dots, n$. We define the *KKT ideal* I_{KKT} , the *KKT varieties*, the *KKT preordering* and the *KKT quadratic module* associated to the KKT system (4.4) as follows.

$$I_{KKT} := \langle L_1, \dots, L_n, \lambda_1 g_1, \dots, \lambda_m g_m \rangle;$$

$$V_{KKT} := \{(x, \lambda) \in \mathbb{C}^n \times \mathbb{C}^m | g(x) = 0 \text{ for all } g \in I_{KKT}\};$$

$$V_{KKT}^{\mathbb{R}} := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m | g(x) = 0 \text{ for all } g \in I_{KKT}\};$$

$$T_{KKT} := T_G + I_{KKT};$$

$$M_{KKT} := M_G + I_{KKT}.$$

² f is called (Khovanskii) *non-degenerate at infinity* if for any face Δ of $\tilde{\Gamma}(f)$ which does not contain the origin $0 \in \mathbb{R}^n$, the system of equations

$$f_{\Delta} = x_1 \frac{\partial f_{\Delta}}{\partial x_1} = \dots = x_n \frac{\partial f_{\Delta}}{\partial x_n} = 0$$

has no solution in $(\mathbb{R} \setminus \{0\})^n$. Here for $f = \sum_{\alpha} f_{\alpha} X^{\alpha} \in \mathbb{R}[X]$, $f_{\Delta} := \sum_{\alpha \in \Delta} f_{\alpha} X^{\alpha}$ denotes the *principal part at infinity* of f with respect to Δ .

Let f_{KKT}^* be the global minimum of f over the KKT system defined by (4.4). Assume the KKT system holds at at least one global minimizer. Then $f^* = f_{KKT}^*$ (cf. [11]). Therefore we have

$$f^* = \min_{x \in V_{KKT}^{\mathbb{R}} \cap K_G} f(x) \quad (4.5)$$

provided that the KKT system holds at at least one global minimizer.

It is well-known (cf. [11]), that if I_{KKT} is zero-dimensional (i.e. V_{KKT} is a finite set) and radical, then $f - f^* \in M_{KKT}$. Moreover, if I_{KKT} is radical, then $f - f^* \in T_{KKT}$ (cf. [11]). For Morse polynomial functions we have

Proposition 4.6. *Let $G = \{g_1, \dots, g_m\}$ be a finite subset of $\mathbb{R}[X]$ such that K_G is compact (resp. M_G is Archimedean). Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume that $V_{KKT}^{\mathbb{R}} \cap K_G$ is finite and contained in the interior of K_G . Then $f - f^* \in T_G$ (resp. $f - f^* \in M_G$).*

Proof. It is obvious that $f - f^*$ is a Morse polynomial function. Moreover, $f - f^* \geq 0$ on K_G , and by assumption, $f - f^*$ vanishes at only finitely many points in the interior of K_G . Then it follows from Theorem 2.2 that $f - f^* \in T_G$ (resp. $f - f^* \in M_G$). \square

To give more applications in polynomial optimization we need to recall some notations (cf. [4]). Let

$$\mathbf{v}_m(\mathbf{x}) := (1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_1x_n, x_2x_3, \dots, x_n^2, \dots, x_1^m, \dots, x_n^m)$$

denotes the canonical basis of the real vector space $\mathbb{R}[X]_m$ of real polynomials of degree at most m , and let $s(m) := \binom{n+m}{m}$ be the dimension of this vector space. If f is a polynomial of degree at most m we may write

$$f = \sum_{\alpha} f_{\alpha} X^{\alpha} = \langle \mathbf{f}, \mathbf{v}_m(\mathbf{x}) \rangle, \text{ where } X^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \sum_{i=1}^n \alpha_i \leq m,$$

and $\mathbf{f} := \{f_{\alpha}\} \in \mathbb{R}^{s(m)}$ denotes the vector of coefficients of f in the basis $\mathbf{v}_m(\mathbf{x})$.

Given an $s(2m)$ -vector $\mathbf{y} := \{y_{\alpha}\}$ with first element $y_{0,\dots,0} = 1$, let $M_m(\mathbf{y})$ be the *moment matrix* of dimension $s(m)$, with rows and columns labeled by the basis $\mathbf{v}_m(\mathbf{x})$.

Let $f \in \mathbb{R}[X]_m$ with coefficient vector $\mathbf{f} \in \mathbb{R}^{s(m)}$. If the entry (i, j) of the matrix $M_m(\mathbf{y})$ is y_{β} , let $\beta(i, j)$ denote the subscript β of y_{β} . We define the matrix $M_m(f\mathbf{y})$ by

$$M_m(f\mathbf{y})(i, j) := \sum_{\alpha} f_{\alpha} y_{\beta(i, j) + \alpha}.$$

Now let $G = \{g_1, \dots, g_m\}$ be a finite subset of $\mathbb{R}[X]$, with each g_i is a polynomial of degree at most w_i . Let $f \in \mathbb{R}[X]_m$ with coefficient vector $\mathbf{f} = \{f_{\alpha}\} \in \mathbb{R}^{s(m)}$. For every $i = 1, \dots, m$, let $\tilde{w}_i = \lceil w_i/2 \rceil$ be the smallest

integer larger than $w_i/2$, and with $N \geq \lceil m/2 \rceil$ and $N \geq \max_i \tilde{w}_i$, consider the convex LMI problem

$$\mathbb{Q}_G^N \begin{cases} \inf_{\mathbf{y}} \sum_{\alpha} f_{\alpha} y_{\alpha}, \\ M_N(\mathbf{y}) \geq 0, \\ M_{N-\tilde{w}_i}(g_i \mathbf{y}) \geq 0, \quad i = 1, \dots, m. \end{cases}$$

Theorem 4.7 ([4, Theorem 4.2]). *Let $G = \{g_1, \dots, g_m\}$ be a finite subset of $\mathbb{R}[X]$ and K_G the basic closed semi-algebraic generated by G . Assume M_G is Archimedean. Let $f \in \mathbb{R}[X]$ be a polynomial of degree m . If there exist a polynomial $q \in \sum \mathbb{R}[X]^2$ of degree at most $2N$ and polynomials $t_i \in \sum \mathbb{R}[X]^2$ of degree at most $2N - w_i$, $i = 1, \dots, m$, such that*

$$f - f^* = q + \sum_{i=1}^m t_i g_i,$$

then $\min \mathbb{Q}_G^N = f^*$, and the vector

$$\mathbf{y}^* := (x_1^*, \dots, x_n^*, (x_1^*)^2, \dots, x_1^* x_2^*, \dots, (x_1^*)^{2N}, \dots, (x_n^*)^{2N})$$

is a global minimizer of \mathbb{Q}_G^N .

Combining Proposition 4.6 and Theorem 4.7, we have the following result.

Corollary 4.8. *Let $G = \{g_1, \dots, g_m\}$ be a finite subset of $\mathbb{R}[X]$ such that M_G is Archimedean. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a Morse polynomial function. Assume that $V_{KKT}^{\mathbb{R}} \cap K_G$ is finite and contained in the interior of K_G . Then there exists a positive integer N such that $\min \mathbb{Q}_G^N = f^*$. Moreover, if $x^* \in V_{KKT}^{\mathbb{R}} \cap K_G$ is a global minimizer of f on K_G , then the vector*

$$\mathbf{y}^* := (x_1^*, \dots, x_n^*, (x_1^*)^2, \dots, x_1^* x_2^*, \dots, (x_1^*)^{2N}, \dots, (x_n^*)^{2N})$$

is a global minimizer of \mathbb{Q}_G^N .

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